Supplement to tutorial 6

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1 What we have covered in the tutorial

In the tutorial, we have discussed

Lemma 1.1. *(Hadamard) Let $U \subset \mathbb{R}^n$, $p = (p_1, p_2, \dots, p_n) \in U$ and $f : U \to \mathbb{R}$ be a smooth function, then there exist smooth functions $g_1, g_2, \dots, g_n : U \to \mathbb{R}$ (possibly replace U by a smaller neighbourhood of p) such that*

$$f(x) = f(p) + \sum_i (x_i - p_i) g_i(x) \tag{1}$$

Proof. Replace U by a smaller open ball centered at p (This makes sure that for $x \in U$, the line $\{p + t(x - p) : t \in [0, 1]\}$ is contained in U). Then we have

$$\begin{aligned} f(x) &= f(p) + \int_0^1 \frac{d}{dt} f(p + t(x - p)) dt \\ &= f(p) + \sum_i (x_i - p_i) \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p)) dt \end{aligned}$$

so we can take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p)) dt$. □

The g_i's are not unique, but if we take the partial derivatives to both sides of equation (1), we know that $g_i(p) = \frac{\partial f}{\partial x_i}(p)$. If we apply the above lemma to g_i, we can then get smooth functions $h_{i1}, h_{i2}, \dots h_{in}$ so that

$$g_i(x) = \frac{\partial f}{\partial x_i}(p) + \sum_j (x_j - p_j) h_{ij}(x).$$

Lemma 1.2. *Let $U \subset \mathbb{R}^n$, $p = (p_1, p_2, \dots, p_n) \in U$ and $f : U \to \mathbb{R}$ be a smooth function, then there exist smooth functions $h_{i,j}, 1 \le i, j \le n$ (possibly replace U by a smaller neighbourhood of p) such that $h_{ij} = h_{ji}$, and*

$$f(x) = f(p) + \sum_i \frac{\partial f}{\partial x_i}(p)(x_i - p_i) + \sum_{i,j} (x_i - p_i)(x_j - p_j) h_{ij}(x)$$

Proof. All follows from the paragraph before the lemma except that we may not have $h_{ij} = h_{ji}$. But we can simply replaced h_{ij} by $\frac{1}{2}(h_{ij} + h_{ji})$. □

In the tutorial, I tried (but not finished) to prove the following,

Lemma 1.3. *Let $U \subset \mathbb{R}^n$, $\overrightarrow{0} \in U$ and $f : U \to \mathbb{R}$ be a smooth function. Suppose $f(\overrightarrow{0}) = 0, DF(\overrightarrow{0})$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}(\overrightarrow{0}) = \delta_{ij}$, then there exists a neighbourhood V of $\overrightarrow{0}$, a smooth map $\Phi : V \to U$ such that $D\Phi(\overrightarrow{0})$ is invertible and*

$$f(\Phi(x_1, x_2, \ldots, x_n)) = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Proof. By lemma 1.2, we can find smooth h_{ij} such that $h_{ij} = h_{ji}$, and

$$f(x) = \sum_{i,j} x_i x_j h_{ij}. \tag{2}$$

By taking the second derivatives to both sides of 1, we found

$$h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\overrightarrow{0}) = \delta_{ij}.$$

We want to do a change of coordinate so that the right hand side becomes the desired form.

Step 1: We first make $h_{12} = h_{21} = 0$.
Since $h_{22}(\overrightarrow{0}) = 1$, we can assume $h_{22} \neq 0$ by restricting to a smaller neighbourhood. Since (completing squares)

$$\begin{aligned} h_{11}x_{11}^2 + x_1x_2h_{12} + x_2x_1h_{21} + x_2^2h_{22} &= h_{11}x_{11}^2 + 2h_{12}x_1x_2 + x_2^2h_{22} \\ &= (h_{11} + h_{22}^{-2}h_{12}^2)x_1^2 + h_{22}(x_2 + h_{22}^{-1}h_{12}x_1)^2 \end{aligned}$$

Consider the function $\Phi_1 : (x_1, x_2, x_3, \ldots, x_n) \mapsto (x_1, x_2 + h_{22}^{-1}h_{12}x_1, x_3 \ldots, x_n)$, then $D\Phi_1(\overrightarrow{0})$ is upper triangular with diagonal entries 1, so Φ_1 has a local inverse by Inverse Function theorem, and so

$$f(\Phi_1^{-1}(x)) = \sum_{i,j} x_i x_j \tilde{h}_{ij},$$

with $\tilde{h}_{ij} = \tilde{h}_{ji}$ and $\tilde{h}_{12} = \tilde{h}_{21} = 0$. To simply the notation, we can replace f by $f \circ \Phi_1^{-1}$, h_{ij} by $\tilde{h}_{ij}$ and assume $h_{12} = h_{21} = 0$.

Step 2: Repeat the above arguments, we can make $h_{13} = h_{31} = 0$, and then $h_{14} = h_{41} = 0$ and so on. So we can assume $h_{1j} = h_{j1} = 0$ for $j > 1$.

Step 3: Repeat step 2, we can make $h_{2j} = hj2 = 0$ for $j > 2$, keep repeating it we can make $h_{ij} = h_{ji} = 0$ for $i \neq j$.

Step 4: Now we have,

$$f(x) = \sum_{i,j} x_i^2 h_{ii},$$

with $h_{ii}(\overrightarrow{0}) = 1$. We can take $\Psi : (x_1, x_2, \ldots, x_n) \mapsto (x_1\sqrt{h_{11}}, x_2\sqrt{h_{22}}, \ldots, x_n\sqrt{h_{nn}})$. Now we can compute and see that $D\Psi$ is the identity matrix, so Ψ has a local inverse Φ, and

$$f(\Phi(x)) = \sum_{i,j} x_i^2$$

□

Actually, using a similar argument, we can prove the following

Lemma 1.4. *Let $U \subset \mathbb{R}^n$, $\overrightarrow{0} \in U$ and $f : U \to \mathbb{R}$ be a smooth function. Suppose $f(\overrightarrow{0}) = 0, DF(\overrightarrow{0})$ and*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\overrightarrow{0}) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \leq k \\ -1 \text{ if } i = j > k \end{cases},$$

then there exists a neighbourhood V of $\overrightarrow{0}$, a smooth map $\Phi : V \to U$ such that $D\Phi(\overrightarrow{0})$ is invertible and

$$f(\Phi(x_1, x_2, \ldots, x_n)) = x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - x_{k+2}^2 + \cdots - x_n^2.$$

Finally,

Theorem 1.5. *(Morse Lemma) Let $U \subset \mathbb{R}^n$, $\overrightarrow{0} \in U$ and $f : U \to \mathbb{R}$ be a smooth function. Suppose $f(\overrightarrow{0}) = 0, DF(\overrightarrow{0})$. Let A be the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p \right)_{i,j}$. Suppose A has signature $(k, n-k)$, then there exists a neighbourhood V of $\overrightarrow{0}$, a smooth map $\Phi : V \to U$ such that $D\Phi(\overrightarrow{0})$ is invertible and*

$$f(\Phi(x_1, x_2, \ldots, x_n)) = x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - x_{k+2}^2 + \cdots - x_n^2.$$

Proof. By Taylor's theorem, we know that

$$f(x) = \sum_{ij} a_{ij} x_i x_j + R(x),$$

where $a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\overrightarrow{0})$, and $R(x) = O(|x|^3)$. Now by linear algebra (symmetric bilinear form), we can do a linear change of variable $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$f(T(x)) = x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - x_{k+2}^2 + \cdots - x_n^2 + R(T(x)).$$

Replacing f with $f \circ T$, we can then use lemma 1.4. □